Resonant Modes of a Concentric Spherical Cavity with Conically Stratified Medium

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***Abstract*—An analysis on electromagnetic fields of a cavity formed by two concentric conducting spheres with a conically stratified medium is presented in this paper. Angular transmission formulation and the radial eigenfunction expansion are used to formulate the field components. Boundary matching methods are applied to obtain the characteristic equations containing various infinite series of spherical Hankel functions and Legendre functions of complex order for resonant frequencies. The first two resonant frequencies and field expansion coefficients are determined numerically. The distribution pattern of angular field components and the forms of typical electric field lines and magnetic field lines for the first resonant fields are also indicated.**

***Index Terms*—Cavity resonators, cones, dielectric devices, electromagnetic fields, mode-matching methods.**

I. Introduction

An ANALYSIS on electromagnetic fields related to a conducting sphere or a multilayered dielectric sphere can be performed analytically since a method of separation of variables can be effectively applied to Maxwell's equation and boundary conditions expressed in the spherical coordinates for spherically stratified regions [1], [2]. It is known that these results are being used well in the field of microwave and lightwave engineering. Similarly, a conducting conical horn is also an electromagnetic wave-guiding system, which is analyzed by the method of separation of variables in the spherical coordinate system. These wave-guiding systems are used as a taper or a conical conducting horn antenna in the microwave engineering field, and their design techniques are also well established [3].

A dielectric circular cone or a dielectric-coated circular horn is also an electromagnetic wave-guiding system. These are also used as a taper or a dielectric antenna in lightwave and microwave engineering. Due to the practical importance of these configurations, they have been subject to numerous investigations [4]–[7].

However, an analytical solution for this wave-guiding system has not yet been found, except for a perturbed approximate solution [8]. Also, a method of solution has been given that employs a number of assumptions and simplifications in order to overcome the mathematical difficulties of this type of problem [9]. Another method has been proposed by using the surface impedance concept to avoid the boundary matching of the field components expressed in the form of the separation of variables in the spherical coordinate system [10], [11].

In fact, in a dielectric conical wave-guiding system, tangential-field components at the conical boundary surface are expressed in terms of spherical Hankel functions with different arguments corresponding to the medium on each side of the boundary. Hence, it is not so simple to match the boundary conditions as in the conducting conical wave-guiding system.

Recently, a method of mode matching for this problem has been proposed, but is not sufficient [12]. Another mode-matching method has also been proposed, which uses the radial eigenfunctions defined in the exterior region of a small conducting sphere at the tip of the dielectric coated conducting cone, where the authors have also overcome the singularity of the Hankel function by placing a small conducting sphere at the tip of the cone [13]. A similar idea has been used to avoid this singularity in the analysis of a dielectric-coated conducting wedge by placing a small conducting cylinder at the tip of the wedge [14].

A resonant cavity, which is formed by two concentric conducting spheres, filled with a conically stratified medium is also an interesting system for the application in microwave and lightwave engineering. However, no literature has been found as far as we know. In this paper, we analyze the electromagnetic fields of this resonant cavity in order to investigate the characteristics of electromagnetic fields in the conically stratified medium. We formulate the boundary-value problem using a standard mode-matching technique.

Assuming that the electromagnetic fields, rotationally symmetric, are independent of azimuthal angle ϕ in the spherical coordinate system ($r,\ \theta,\ \phi$), we derive transmission equations of electromagnetic fields in the direction of polar angle θ. Next, we introduce radial expansion eigenfunctions of two types in each dielectric medium subject to the boundary conditions on the two conducting spheres. The first type is the transverse-electric type to θ (TE-to-θ, H type) and the other is transverse-magnetic type to θ (TM-to-θ, E type). Expanding the electromagnetic fields by the expansion eigenfunctions, we derive the θ transmission-line equations in each dielectric medium. Determining the field components and applying boundary conditions at the interface of dielectric cones, we derive the characteristic equations to determine the resonant frequency of the cavity. Analyzing numerically the characteristic equations, we obtain resonant eigenfrequencies and electromagnetic eigenfields. The distribution pattern of angular field components are shown. The forms of the typical field lines for first resonant frequency of both TM and TE fields are also indicated.

II. Formulation of the Electromagnetic Fields

A. E- and H-Type Field Equations

The geometry of a cavity resonator, analyzed here, is shown in Fig. 1. We use the spherical coordinate system for our analysis,

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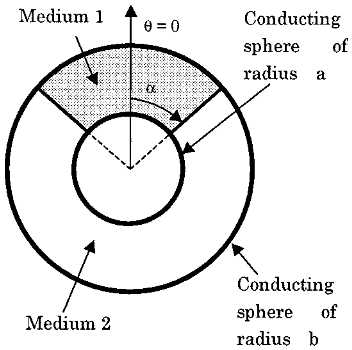


Fig. 1. Schematic diagram of a concentric spherical cavity with conically stratified dielectric medium.

as shown in Fig. 1. The cavity consists of two concentric perfectly conducting spheres. The region bound by two spheres is filled with a conically stratified dielectric medium. In the spherical coordinate system, the phasor form of Maxwell's equations in a source-free region are given as follows:

$$\frac{1}{r\sin\theta}\frac{\partial(\sin\theta H_\phi)}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial(H_\theta)}{\partial\phi}=j\omega\varepsilon E_r \tag{1}$$

$$\frac{1}{r\sin\theta}\frac{\partial(H_r)}{\partial\phi}-\frac{1}{r}\frac{\partial(rH_\phi)}{\partial r}=j\omega\varepsilon E_\theta \tag{2}$$

$$\frac{1}{r}\frac{\partial(rH_\theta)}{\partial r}-\frac{1}{r}\frac{\partial(H_r)}{\partial\theta}=j\omega\varepsilon E_\phi \tag{3}$$

$$\frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\phi)}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial(E_\theta)}{\partial\phi}=-j\omega\mu H_r \tag{4}$$

$$\frac{1}{r\sin\theta}\frac{\partial(E_r)}{\partial\phi}-\frac{1}{r}\frac{\partial(rE_\phi)}{\partial r}=-j\omega\mu H_\theta \tag{5}$$

$$\frac{1}{r}\frac{\partial(rE_\theta)}{\partial r}-\frac{1}{r}\frac{\partial(E_r)}{\partial\theta}=-j\omega\mu H_\phi. \tag{6}$$

We assume that $\mu=\mu_0$ and $\varepsilon=\varepsilon_0\varepsilon_1$ in medium 1 ($0\le\theta<\alpha$) and $\mu=\mu_0$ and $\varepsilon=\varepsilon_0\varepsilon_2$ in medium 2 ($\alpha\le\theta\le\pi$). The time dependence $e^{j\omega t}$ is assumed and suppressed throughout. Assuming that the fields are independent of azimuthal angle ϕ, we can separate equations from (1)–(6) into two groups as follows:

E-type (TM-to-θ)

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial(rE_\theta)}{\partial r}-\frac{1}{r}\frac{\partial(E_r)}{\partial\theta}&=-j\omega\mu H_\phi\\ \frac{1}{r\sin\theta}\frac{\partial(\sin\theta H_\phi)}{\partial\theta}&=j\omega\varepsilon E_r\\ \frac{1}{r}\frac{\partial(rH_\phi)}{\partial r}&=-j\omega\varepsilon E_\theta\end{aligned}\right\} \tag{7}$$

H-type (TE-to-θ)

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial(rH_\theta)}{\partial r}-\frac{1}{r}\frac{\partial(H_r)}{\partial\theta}&=j\omega\varepsilon E_\phi\\ \frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\phi)}{\partial\theta}&=-j\omega\mu H_r\\ \frac{1}{r}\frac{\partial(rE_\phi)}{\partial r}&=j\omega\mu H_\theta\end{aligned}\right\}. \tag{8}$$

The field components of E type are H_ϕ, E_r, and E_θ, and those of H type are E_ϕ,H_r and H_θ. From (7), we obtain

$$\left.\begin{aligned}\frac{\partial(r^2E_r)}{\partial\theta}&=j\omega\mu r^2\left(rH_\phi+\frac{1}{\omega^2\mu\varepsilon}\frac{\partial^2(rH_\phi)}{\partial r^2}\right)\\ \frac{1}{\sin\theta}\frac{\partial(r\sin\theta H_\phi)}{\partial\theta}&=jr^2\omega\varepsilon E_r\end{aligned}\right\} \tag{9}$$

where (9) is the θ transmission-field equations of the TM field [15]. We also obtain from (8) the following:

$$\left.\begin{aligned}\frac{\partial(r^2H_r)}{\partial\theta}&=-j\omega\varepsilon r^2\left(rE_\phi+\frac{1}{\omega^2\mu\varepsilon}\frac{\partial^2(rE_\phi)}{\partial r^2}\right)\\ \frac{1}{\sin\theta}\frac{\partial(r\sin\theta E_\phi)}{\partial\theta}&=-jr^2\omega\mu H_r\end{aligned}\right\}. \tag{10}$$

Equation (10) is the θ transmission-field equations of the TE field [15].

B. Expansion Eigenfunctions

Here, we introduce the eigenfunctions.

1) E-Type Eigenfunctions: e_i^E and h_i^E ($i=1,2,3,\ldots$):

$$\frac{\partial^2(h_i^E)}{\partial\rho^2}+\left(1-\frac{\kappa_i(\kappa_i+1)}{\rho^2}\right)h_i^E=0,\qquad \rho_a\le\rho\le\rho_b \tag{11}$$

$$\left.\frac{\partial h_i^E}{\partial\rho}\right|_{\substack{\rho=\rho_a\\ \rho=\rho_b}}=0 \tag{12}$$

where $\rho=\sqrt{\varepsilon_s}\,k_0r\triangleq\sqrt{\varepsilon_s}\,\rho_0$, $k_0=\omega\sqrt{\mu_0\varepsilon_0}$, $\rho_a=k_0\sqrt{\varepsilon_s}\,a$, $\rho_b=k_0\sqrt{\varepsilon_s}\,b$, and $s=1,2$. We assume that the eigenfunctions h_i^E ($i=1,2,3,\ldots$) satisfy the following orthonormal relation:

$$\int_{\rho_a}^{\rho_b}\frac{h_i^E(\rho)h_j^E(\rho)}{\rho^2}\,d\rho=\delta_{ij} \tag{13}$$

where δ_{ij} is Kronecker's delta. We write a solution of (11) as follows:

$$h_i^E=p_i^E\left[\hat{H}_{\kappa_i}^{(1)}(\rho)+\lambda_i^E\hat{H}_{\kappa_i}^{(2)}(\rho)\right] \tag{14}$$

where $\hat{H}_{\kappa_i}^{(1),(2)}(\rho)$ is the spherical Hankel function used by Debye and Schelkunoff [16]. It is related to the Hankel function by $\hat{H}_{\kappa_i}^{(1)/(2)}(\rho)=(\pi\rho/2)^{1\backslash 2}H_{\kappa_i+1/2}^{(1),(2)}(\rho)$. The constant p_i^E is determined by the normalization condition as follows:

$$(p_i^E)^2=\frac{1}{\displaystyle\int_{\rho_a}^{\rho_b}\frac{\left[\hat{H}_{\kappa_i}^{(1)}(\rho)+\lambda_i^E\hat{H}_{\kappa_i}^{(2)}(\rho)\right]^2}{\rho^2}\,d\rho} \tag{15}$$

and the constant λ_i^E is determined by the boundary conditions as follows:

$$\frac{\hat{H}_{\kappa_i}^{(1)\prime}(\rho_a)}{\hat{H}_{\kappa_i}^{(2)\prime}(\rho_a)}=\frac{\hat{H}_{\kappa_i}^{(1)\prime}(\rho_b)}{\hat{H}_{\kappa_i}^{(2)\prime}(\rho_b)}=-\lambda_i^E \tag{16}$$

where the prime ($'$) denotes the differentiation with respect to its argument.

We further define e_i^E as follows:

$$e_i^E=-h_i^E. \tag{17}$$

2) H-Type Eigenfunctions: e_i^H and $h_i^H (i = 1, 2, 3, \ldots)$:

$$\frac{\partial^2(e_i^H)}{\partial\rho^2}+\left(1-\frac{\nu_i(\nu_i+1)}{\rho^2}\right)e_i^H = 0, \qquad \rho_a \le \rho \le \rho_b \tag{18}$$

$$e_i^H\Big|_{\substack{\rho=\rho_a\\ \rho=\rho_b}} = 0 \tag{19}$$

$$\int_{\rho_a}^{\rho_b} \frac{e_i^H(\rho)e_j^H(\rho)}{\rho^2}\,d\rho = \delta_{ij} \quad \text{(orthonormal relation)} \tag{20}$$

$$e_i^H = p_i^H\left[\hat{H}_{\nu_i}^{(1)}(\rho)+\lambda_i^H\hat{H}_{\nu_i}^{(2)}(\rho)\right] \tag{21}$$

$$(p_i^H)^2 = \frac{1}{\displaystyle\int_{\rho_a}^{\rho_b}\frac{\left[\hat{H}_{\nu_i}^{(1)}(\rho)+\lambda_i^H\hat{H}_{\nu_i}^{(2)}(\rho)\right]^2}{\rho^2}\,d\rho} \tag{22}$$

and the constant λ_i^H is determined by the boundary conditions as follows:

$$\frac{\hat{H}_{\nu_i}^{(1)}(\rho_a)}{\hat{H}_{\nu_i}^{(2)}(\rho_a)} = \frac{\hat{H}_{\nu_i}^{(1)}(\rho_b)}{\hat{H}_{\nu_i}^{(2)}(\rho_b)} = -\lambda_i^H. \tag{23}$$

We further define h_i^H as follows:

$$h_i^H = e_i^H. \tag{24}$$

C. θ-Transmission Formulations

We expand the field by E-type eigenfunctions $e_i^E$and h_i^E. In (9), assuming that

$$\left.\begin{aligned} r^2E_r &= \sum_i V_i^E(\theta)e_i^E(\rho)\\ rH_\phi &= \sum_i \frac{1}{\sin\theta}I_i^E(\theta)h_i^E(\rho)\end{aligned}\right\} \tag{25}$$

and applying the orthonormal relation (13), we obtain

$$\left.\begin{aligned} \sin\theta\frac{\partial(V_i^E)}{\partial\theta} &= -\frac{j\kappa_i(\kappa_i+1)}{\omega\varepsilon}I_i^E\\ \frac{\partial(I_i^E)}{\partial\theta} &= -j\omega\varepsilon\sin\theta V_i^E\end{aligned}\right\}. \tag{26}$$

Equation (26) is the transmission-line equation in the θ-direction for TM fields. Eliminating I_i^E from (26), we obtain

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial(V_i^E)}{\partial\theta}\right)+\kappa_i(\kappa_i+1)V_i^E = 0. \tag{27}$$

Solutions of (27) are expressed by the Legendre function $P_{\kappa_i}(\cos\theta)$ and $P_{\kappa_i}(-\cos\theta)$ as follows:

$$V_i^E = \begin{cases} C_i^{(1)}P_{\kappa_i}^{(1)}(\cos\theta), & 0\le\theta<\alpha\\ C_i^{(2)}P_{\kappa_i}^{(2)}(-\cos\theta), & \alpha\le\theta\le\pi.\end{cases} \tag{28}$$

Similarly, in (10), assuming that

$$\left.\begin{aligned} r^2H_r &= \sum_i I_i^H(\theta)h_i^H(\rho)\\ rE_\phi &= \sum_i \frac{1}{\sin\theta}V_i^H(\theta)e_i^H(\rho)\end{aligned}\right\} \tag{29}$$

and using the orthonormal relation (20), we obtain

$$\left.\begin{aligned} \sin\theta\frac{\partial(I_i^H)}{\partial\theta} &= -\frac{j\nu_i(\nu_i+1)}{\omega\mu}V_i^H\\ \frac{\partial(V_i^H)}{\partial\theta} &= -j\omega\mu\sin\theta I_i^H\end{aligned}\right\}. \tag{30}$$

Equation (30) is the transmission-line equation in the θ-direction for TE fields. Eliminating V_i^H from (30), we obtain

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial(I_i^H)}{\partial\theta}\right)+\nu_i(\nu_i+1)I_i^H = 0. \tag{31}$$

Solutions of (31) are also expressed by the Legendre function as follows:

$$I_i^H = \begin{cases} D_i^{(1)}P_{\nu_i}^{(1)}(\cos\theta), & 0\le\theta<\alpha\\ D_i^{(2)}P_{\nu_i}^{(2)}(-\cos\theta), & \alpha\le\theta\le\pi.\end{cases} \tag{32}$$

Finally, we express the electromagnetic fields in mediums 1 and 2 by a series expansion of orthonormal functions as follows.:

For E-fields in medium 1

$$E_r^{(1)} = -\sum_i \frac{C_i^{(1)}P_{\kappa_i^{(1)}}(\cos\theta)}{r^2}U_{\kappa_i^{(1)}}^E\left(\sqrt{\varepsilon_1}\,\rho_0\right) \tag{33}$$

$$E_\theta^{(1)} = -\sum_i \frac{k_0\sqrt{\varepsilon_1}\,C_i^{(1)}\frac{\partial}{\partial\theta}P_{\kappa_i^{(1)}}(\cos\theta)}{r\kappa_i^{(1)}\left(\kappa_i^{(1)}+1\right)}U_{\kappa_i^{(1)}}^{E'}\left(\sqrt{\varepsilon_1}\,\rho_0\right) \tag{34}$$

$$H_\phi^{(1)} = \sum_i \frac{j\omega\varepsilon_0\varepsilon_1C_i^{(1)}\frac{\partial}{\partial\theta}P_{\kappa_i^{(1)}}(\cos\theta)}{r\kappa_i^{(1)}\left(\kappa_i^{(1)}+1\right)}U_{\kappa_i^{(1)}}^E\left(\sqrt{\varepsilon_1}\,\rho_0\right). \tag{35}$$

For H-fields in medium 1

$$H_r^{(1)} = \sum_i \frac{D_i^{(1)}P_{\nu_i^{(1)}}(\cos\theta)}{r^2}U_{\nu_i^{(1)}}^H\left(\sqrt{\varepsilon_1}\,\rho_0\right) \tag{36}$$

$$H_\theta^{(1)} = \sum_i \frac{k_0\sqrt{\varepsilon_1}\,D_i^{(1)}\frac{\partial}{\partial\theta}P_{\nu_i^{(1)}}(\cos\theta)}{r\nu_i^{(1)}\left(\nu_i^{(1)}+1\right)}U_{\nu_i^{(1)}}^{H'}\left(\sqrt{\varepsilon_1}\,\rho_0\right) \tag{37}$$

$$E_\phi^{(1)} = \sum_i \frac{j\omega\mu_0D_i^{(1)}\frac{\partial}{\partial\theta}P_{\nu_i^{(1)}}(\cos\theta)}{r\nu_i^{(1)}\left(\nu_i^{(1)}+1\right)}U_{\nu_i^{(1)}}^H\left(\sqrt{\varepsilon_1}\,\rho_0\right). \tag{38}$$

For E-fields in medium 2

$$E_r^{(2)} = -\sum_i \frac{C_i^{(2)}P_{\kappa_i^{(2)}}(-\cos\theta)}{r^2}U_{\kappa_i^{(2)}}^E\left(\sqrt{\varepsilon_2}\rho_0\right) \tag{39}$$

$$E_\theta^{(2)} = -\sum_i \frac{k_0\sqrt{\varepsilon_2}\, C_i^{(2)} \frac{\partial}{\partial\theta} P_{\kappa_i^{(2)}}(-\cos\theta)}{r\kappa_i^{(2)}\left(\kappa_i^{(2)}+1\right)} U^{E'}_{\kappa_i^{(2)}}\left(\sqrt{\varepsilon_2}\,\rho_0\right) \tag{40}$$

$$H_\phi^{(2)} = \sum_i \frac{j\omega\varepsilon_0\varepsilon_2 C_i^{(2)} \frac{\partial}{\partial\theta} P_{\kappa_i^{(2)}}(-\cos\theta)}{r\kappa_i^{(2)}\left(\kappa_i^{(2)}+1\right)} U^{E}_{\kappa_i^{(2)}}\left(\sqrt{\varepsilon_2}\,\rho_0\right). \tag{41}$$

For H-fields in medium 2

$$H_r^{(2)} = \sum_i \frac{D_i^{(2)} P_{\nu_i^{(2)}}(-\cos\theta)}{r^2} U^H_{\nu_i^{(2)}}\left(\sqrt{\varepsilon_2}\,\rho_0\right) \tag{42}$$

$$H_\theta^{(2)} = \sum_i \frac{k_0\sqrt{\varepsilon_2}\, D_i^{(2)} \frac{\partial}{\partial\theta} P_{\nu_i^{(2)}}(-\cos\theta)}{r\nu_i^{(2)}\left(\nu_i^{(2)}+1\right)} U^{H'}_{\nu_i^{(2)}}\left(\sqrt{\varepsilon_2}\,\rho_0\right) \tag{43}$$

$$E_\phi^{(2)} = \sum_i \frac{j\omega\mu_0 D_i^{(2)} \frac{\partial}{\partial\theta} P_{\nu_i^{(2)}}(-\cos\theta)}{r\nu_i^{(2)}\left(\nu_i^{(2)}+1\right)} U^{H}_{\nu_i^{(2)}}\left(\sqrt{\varepsilon_2}\,\rho_0\right) \tag{44}$$

where

$$U^E_{\kappa_i^{(s)}}\left(\sqrt{\varepsilon_s}\,\rho_0\right) = p_i^{E^{(s)}}\left[\hat{H}^{(1)}_{\kappa_i^{(s)}}\left(\sqrt{\varepsilon_s}\,\rho_0\right) + \lambda_i^{E^{(s)}} \hat{H}^{(2)}_{\kappa_i^{(s)}}\left(\sqrt{\varepsilon_s}\,\rho_0\right)\right], \quad s=1,2 \tag{45}$$

$$U^H_{\nu_i^{(s)}}\left(\sqrt{\varepsilon_s}\,\rho_0\right) = p_i^{H^{(s)}}\left[\hat{H}^{(1)}_{\nu_i^{(s)}}\left(\sqrt{\varepsilon_s}\,\rho_0\right) + \lambda_i^{H^{(s)}} \hat{H}^{(2)}_{\nu_i^{(s)}}\left(\sqrt{\varepsilon_s}\,\rho_0\right)\right], \quad s=1,2 \tag{46}$$

and $U_\kappa^{E'}$ and $U_\nu^{H'}$ denote derivatives of U_κ^E and U_ν^H with respect to their argument, respectively.

III. CHARACTERSTIC EQUATIONS

At the conical surface boundary of two media, the tangential component of the electromagnetic field must be continuous. Thus, we have

$$E_r\big|_{\substack{\varepsilon_r=\varepsilon_1\\ \theta=\alpha-0}} = E_r\big|_{\substack{\varepsilon_r=\varepsilon_2\\ \theta=\alpha+0}} \tag{47}$$

$$H_\phi\big|_{\substack{\varepsilon_r=\varepsilon_1\\ \theta=\alpha-0}} = H_\phi\big|_{\substack{\varepsilon_r=\varepsilon_2\\ \theta=\alpha+0}} \tag{48}$$

$$H_r\big|_{\substack{\varepsilon_r=\varepsilon_1\\ \theta=\alpha-0}} = H_r\big|_{\substack{\varepsilon_r=\varepsilon_2\\ \theta=\alpha+0}} \tag{49}$$

and

$$E_\phi\big|_{\substack{\varepsilon_r=\varepsilon_1\\ \theta=\alpha-0}} = E_\phi\big|_{\substack{\varepsilon_r=\varepsilon_2\\ \theta=\alpha+0}}. \tag{50}$$

From (33), (39), and (47), we obtain the following:

$$\sum_i \frac{C_i^{(1)} P_{\kappa_i^{(1)}}(\cos\alpha)}{r^2} U^E_{\kappa_i^{(1)}}\left(\sqrt{\varepsilon_1}\,\rho_0\right) = \sum_i \frac{C_i^{(2)} P_{\kappa_i^{(2)}}(-\cos\alpha)}{r^2} U^E_{\kappa_i^{(2)}}\left(\sqrt{\varepsilon_2}\,\rho_0\right). \tag{51}$$

Multiplying both sides of (51) by $U^E_{\kappa_j^{(1)}}\left(\sqrt{\varepsilon_1}\rho_0\right)$ and integrating from $r=a$ to $r=b$, we obtain

$$C_j^{(1)} = \sum_i \frac{C_i^{(2)}}{\sqrt{\varepsilon_1}} \frac{P_{\kappa_i^{(2)}}(-\cos\alpha)}{P_{\kappa_j^{(1)}}(\cos\alpha)} F_\kappa^E(j,i). \tag{52}$$

Similarly, we obtain from (35), (41), and (48)

$$C_j^{(1)} = \left(\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_i \frac{C_i^{(2)}}{\sqrt{\varepsilon_1}} \frac{\dfrac{\frac{\partial}{\partial\theta} P_{\kappa_i^{(2)}}(-\cos\theta)\Big|_{\theta=\alpha}}{\kappa_i^{(2)}(\kappa_i^{(2)}+1)}}{\dfrac{\frac{\partial}{\partial\theta} P_{\kappa_j^{(1)}}(\cos\theta)\Big|_{\theta=\alpha}}{\kappa_j^{(1)}(\kappa_j^{(1)}+1)}} F_\kappa^E(j,i). \tag{53}$$

Further, from (36), (42), and (49)

$$D_j^{(1)} = \sum_i \frac{D_i^{(2)}}{\sqrt{\varepsilon_1}} \frac{P_{\nu_i^{(2)}}(-\cos\alpha)}{P_{\nu_j^{(1)}}(\cos\alpha)} F_\nu^H(j,i) \tag{54}$$

and from (38), (44), and (50)

$$D_j^{(1)} = \sum_i \frac{D_i^{(2)}}{\sqrt{\varepsilon_1}} \frac{\dfrac{\frac{\partial}{\partial\theta} P_{\nu_i^{(2)}}(-\cos\theta)\Big|_{\theta=\alpha}}{\nu_i^{(2)}(\nu_i^{(2)}+1)}}{\dfrac{\frac{\partial}{\partial\theta} P_{\nu_j^{(1)}}(\cos\theta)\Big|_{\theta=\alpha}}{\nu_j^{(1)}(\nu_j^{(1)}+1)}} F_\nu^H(j,i) \tag{55}$$

where

$$F_\kappa^E(j,i) = \int_{k_0 a}^{k_0 b} \frac{U^E_{\kappa_i^{(2)}}\left(\sqrt{\varepsilon_2}\,\rho_0\right) U^E_{\kappa_j^{(1)}}\left(\sqrt{\varepsilon_1}\,\rho_0\right)}{\rho_0^2}\, d\rho_0 \tag{56}$$

$$F_\nu^H(j,i) = \int_{k_0 a}^{k_0 b} \frac{U^H_{\nu_i^{(2)}}\left(\sqrt{\varepsilon_2}\,\rho_0\right) U^H_{\nu_j^{(1)}}\left(\sqrt{\varepsilon_1}\,\rho_0\right)}{\rho_0^2}\, d\rho_0. \tag{57}$$

We simplify the pair of equations (52) and (53) as follows:

$$\sum_i C_i^{(2)} P_{\kappa_i^{(2)}}(-\cos\alpha) F_\kappa^E(j,i) G_\kappa^E(j,i) = 0, \quad j=1,2,3,\ldots. \tag{58}$$

Similarly the pair of equations (54) and (55) can be simplified as follows:

$$\sum_i D_i^{(2)} P_{\nu_i^{(2)}}(-\cos\alpha) F_\nu^H(j,i) G_\nu^H(j,i) = 0, \quad j=1,2,3,\ldots \tag{59}$$

where

$$G_\kappa^E(j,i) = \left\{1 - \frac{\varepsilon_2 \dfrac{\cos\alpha P_{\kappa_i^{(2)}}(-\cos\alpha) + P_{\kappa_i^{(2)}-1}(-\cos\alpha)}{(\kappa_i^{(2)}+1) P_{\kappa_i^{(2)}}(-\cos\alpha)}}{\varepsilon_1 \dfrac{\cos\alpha P_{\kappa_j^{(1)}}(\cos\alpha) - P_{\kappa_j^{(1)}-1}(\cos\alpha)}{(\kappa_j^{(1)}+1) P_{\kappa_j^{(1)}}(\cos\alpha)}}\right\} \tag{60}$$

$$G_\nu^H(j,i) = \left\{ 1 - \frac{\dfrac{\cos\alpha P_{\nu_i^{(2)}}(-\cos\alpha) + P_{\nu_i^{(2)}-1}(-\cos\alpha)}{(\nu_i^{(2)}+1)P_{\nu_i^{(2)}}(-\cos\alpha)}}{\dfrac{\cos\alpha P_{\nu_j^{(1)}}(\cos\alpha) - P_{\nu_j^{(1)}-1}(\cos\alpha)}{(\nu_j^{(1)}+1)P_{\nu_j^{(1)}}(\cos\alpha)}} \right\}. \quad (61)$$

By truncating the number of terms according to the required accuracy, we write (58) in a finite dimensional matrix form

$$R \cdot X = [r_{pq}]_{N\times N} \cdot \left[x_p^{(2)}\right]_{N\times 1} = [0]_{N\times 1} \quad (62)$$

where

$$x_p^{(2)} = C_p^{(2)} P_{\kappa_p^{(2)}}(-\cos\alpha) \quad (63)$$

and

$$r_{pq} = F_\kappa^E(p,q) G_\kappa^E(p,q). \quad (64)$$

Similarly, (59) can also be written in the finite dimensional matrix form

$$S \cdot Y = [s_{pq}]_{N\times N} \cdot \left[y_p^{(2)}\right]_{N\times 1} = [0]_{N\times 1} \quad (65)$$

where

$$y_p^{(2)} = D_p^{(2)} P_{\nu_p^{(2)}}(-\cos\alpha) \quad (66)$$

and

$$s_{pq} = F_\nu^H(p,q) G_\nu^H(p,q). \quad (67)$$

From the condition for the nontrivial solution of X and Y, the determinant of matrix R and S must be zero as follows:

$$\det[R] = 0 \quad (68)$$

and

$$\det[S] = 0. \quad (69)$$

Equation (68) associated with (16) and (69) associated with (23) are the characteristic equations for the resonant modes of the cavity. Analyzing (68) and (16) numerically, we determine the resonant frequencies of TM fields of this cavity. Similarly, from (69) and (23), the resonant frequencies of the TE field are determined. Once we find the resonant frequencies, we can obtain the field expansion coefficients.

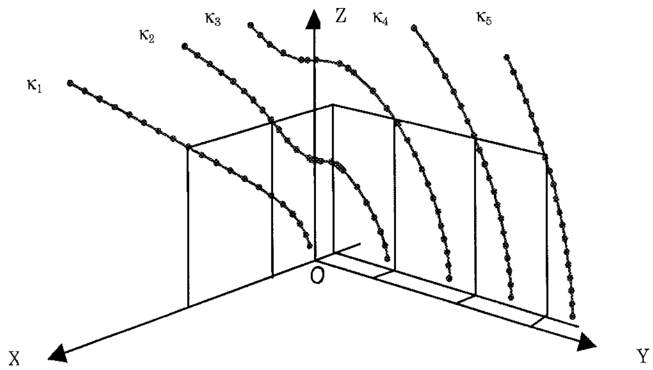


Fig. 2. Solutions of $\hat{H}_{\kappa_i}^{(1)\prime}(Z)\hat{H}_{\kappa_i}^{(2)\prime}(Zb/a) - \hat{H}_{\kappa_i}^{(1)\prime}(Zb/a)\hat{H}_{\kappa_i}^{(2)\prime}(Z) = 0$ (for the TM field) when $b/a = 5$. X-, Y-, and Z-axes represent real κ_i, imaginary κ_i, and $\sqrt{\varepsilon_r}\, k_0 a$, respectively.

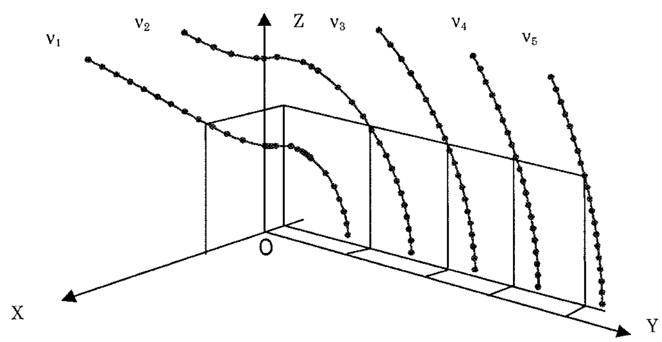


Fig. 3. Solutions of $\hat{H}_{\nu_i}^{(1)}(Z)\hat{H}_{\nu_i}^{(2)}(Zb/a) - \hat{H}_{\nu_i}^{(1)}(Zb/a)\hat{H}_{\nu_i}^{(2)}(Z) = 0$ (for the TE field) when $b/a = 5$. X-, Y-, and Z-axes represent real ν_i, imaginary ν_i and $\sqrt{\varepsilon_r}\, k_0 a$, respectively.

IV. Evaluation and Results

A. Resonant Frequencies and Field Expansion Coefficients

Rewriting (16) and (23) for the convenience of computation, we have

$$\hat{H}_{\kappa_i}^{(1)\prime}(Z)\hat{H}_{\kappa_i}^{(2)\prime}(Zb/a) - \hat{H}_{\kappa_i}^{(1)\prime}(Zb/a)\hat{H}_{\kappa_i}^{(2)\prime}(Z) = 0 \quad (70)$$

for the TM field and

$$\hat{H}_{\nu_i}^{(1)}(Z)\hat{H}_{\nu_i}^{(2)}(Zb/a) - \hat{H}_{\nu_i}^{(1)}(Zb/a)\hat{H}_{\nu_i}^{(2)}(Z) = 0 \quad (71)$$

for the TE field. Here, we assume that

$$Z = \begin{cases} k_0 a\sqrt{\varepsilon_1}, & \text{for medium 1} \\ k_0 a\sqrt{\varepsilon_2}, & \text{for medium 2.} \end{cases} \quad (72)$$

We numerically analyze (68) and (70) for the TM field and (69) and (71) for the TE field. For convenience, a graph of κ_i as a function of Z determined from (70) and a graph of ν_i as a function of Z determined from (71) are shown in Figs. 2 and 3, respectively. We also refer to the literature for the computation of specific functions [17]–[20].

TABLE I
RESONANT FREQUENCIES WITH THE NUMBER (N) OF EXPANSION EIGENFUNCTIONS TAKEN FOR COMPUTATION

N	TM-field resonant frequencies		TE-field resonant frequencies	
	$(k_0a)^E_{\omega_1}$	$(k_0a)^E_{\omega_2}$	$(k_0a)^H_{\omega_1}$	$(k_0a)^H_{\omega_2}$
5	0.40584260	0.61057065	0.69241738	0.95895273
10	0.40584499	0.61057558	0.69242090	0.95895321
15	0.40584501	0.61057563	0.69242092	0.95895321

TABLE II
FIRST 15 FIELD EXPANSION COEFFICIENTS IN EACH MEDIUM FOR THE FIRST RESONANT FREQUENCY OF TM AND TE FIELDS

i	Expansion-coefficients at the first resonant frequency of TM-field		Expansion-coefficients at the first resonant frequency of TE-field	
	$\left(\frac{x_i^{(1)}}{x_1^{(2)}}\right)_{\omega_1}$	$\left(\frac{x_i^{(2)}}{x_1^{(2)}}\right)_{\omega_1}$	$\left(\frac{y_i^{(1)}}{y_1^{(2)}}\right)_{\omega_1}$	$\left(\frac{y_i^{(2)}}{y_1^{(2)}}\right)_{\omega_1}$
1	0.395989	1.000000	0.963629	1.000000
2	0.675851	-0.458736	-0.404629	0.986569
3	0.113814	0.198458	-0.109036	-0.445308
4	-0.048950	-0.085667	0.048778	0.185946
5	0.020983	0.037612	0.023115	-0.080325
6	-0.012509	-0.020611	-0.012771	0.038440
7	0.006810	-0.011179	0.007271	-0.019811
8	-0.004874	0.007540	-0.004583	-0.011249
9	0.002907	0.004548	0.002920	0.006706
10	0.002423	0.003608	-0.002009	-0.004307
11	-0.001436	-0.002200	0.001380	-0.002837
12	0.001427	-0.002072	-0.001009	-0.001982
13	0.000758	-0.001158	0.000732	0.001400
14	-0.000959	-0.001371	0.000560	0.001041
15	-0.000402	-0.000643	-0.000423	0.000803

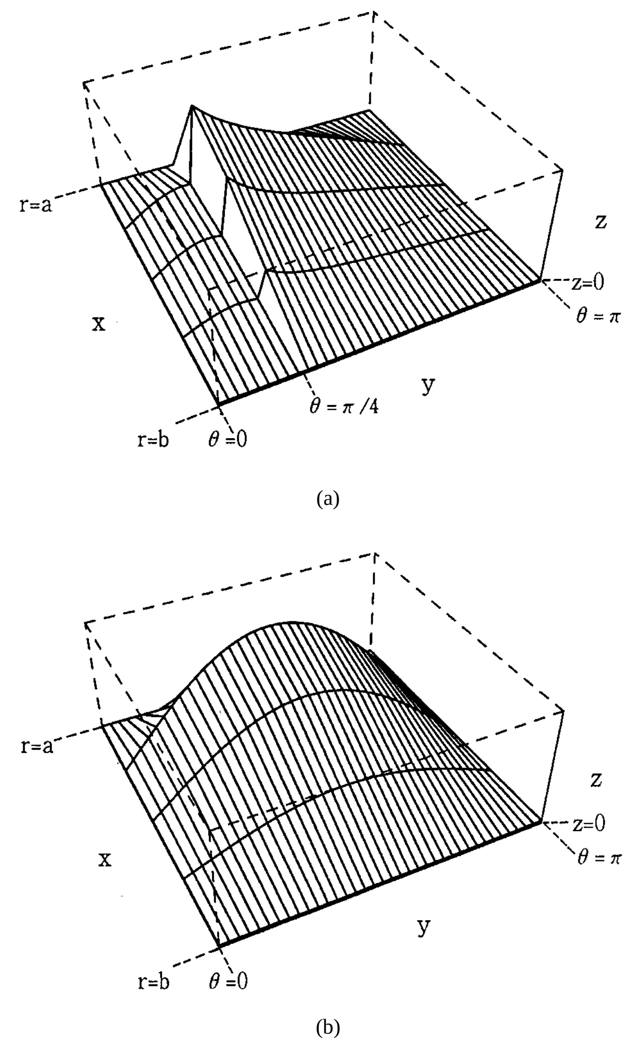


Fig. 4. (a) Field-distribution pattern of E_θ for the first resonant frequency ($k_0a = 0.40584501$) of a concentric spherical cavity with conically stratified medium when $b/a = 5, \varepsilon_1 = 4, \varepsilon_2 = 1$, and $\alpha = \pi/4$. (b) Field-distribution pattern of E_θ for the first resonant frequency ($k_0a = 0.52397273$) of a concentric spherical cavity with homogenous medium when $b/a = 5$ and $\varepsilon_r = 1$. Throughout, x-, y-, and z-axes represent r, θ, and E_θ (relative value), respectively.

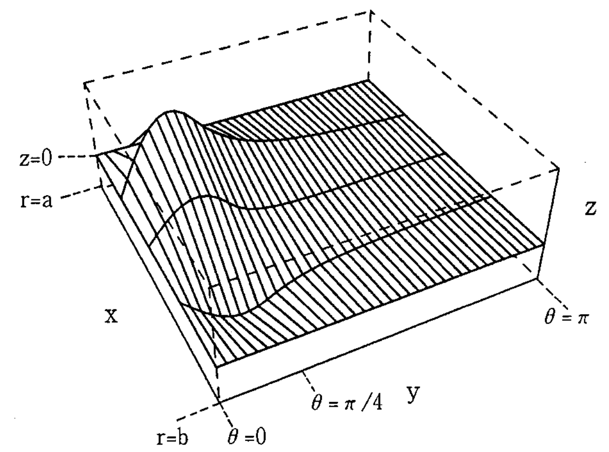
For a given k_0a, we calculate the (68) via (70) for the TM field and (69) via (71) for the TE field. In computation, κ_i and ν_i are calculated to at least 12 digits of accuracy for a given k_0a. We calculate the resonant frequencies by taking first five, ten, and 15 expansion eigenfunctions. Table I shows the resonant frequencies with the number (N) of expansion eigenfunctions taken into account for the computation. It is clear from the results that by taking 15 expansion eigenfunctions, we can calculate the resonant frequencies to at least seven digits of accuracy.

Finally, the field expansion coefficients are also calculated at the first resonant frequencies of the TM and TE fields, and the results are shown in Table II, which shows that the expansion coefficients are converging to zero. By taking 15 expansion eigenfunctions, we can evaluate the field components to at least three digits of accuracy.

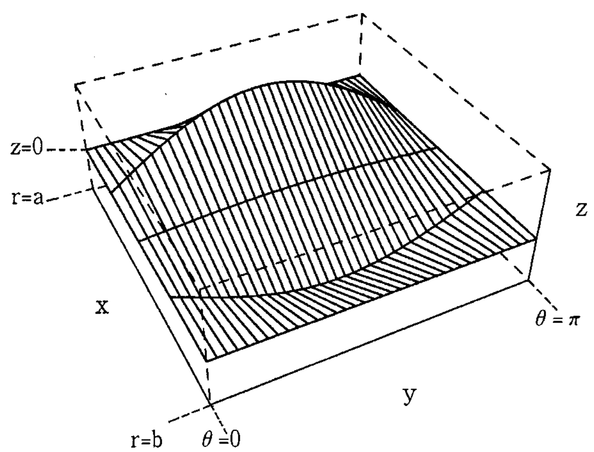
B. Field Distributions and Field Lines

We three-dimensionally show the distribution pattern of the electric-field component E_θ for TM fields and the magnetic-field component H_θ for TE fields on the ϕ coordinate surface in the cavity both with a conically stratified medium and a homogeneous medium in Figs. 4 and 5, respectively. Figs. 4(a) and 5(a) show that the electromagnetic fields are bound by a conical dielectric and the fields outside the conical dielectric are evanescent toward the surface $\theta = \pi$. These results are understood from the fact that the resonator of a circular dielectric rod shielded by two conducting plates is a limiting form of the cavity resonator treated here [21].

We show the forms of typical field lines of both the TM and TE fields for the first resonant frequencies. For comparison, the electric field lines of the TM fields of the concentric spherical cavity both with a conically stratified medium and with a homogeneous medium are shown in Fig. 6. Similarly, the magnetic field lines of the TE fields of the concentric spherical cavity both with a conically stratified medium and with a homogeneous medium are also shown in Fig. 7. The broken lines in Fig. 6 show the locus of the points where $E_r = 0$. Also, the broken lines in Fig. 7 show the locus of the points where $H_r = 0$. The broken lines in Fig. 6 (where $E_r = 0$) and Fig. 7 (where $H_r = 0$) show that the bounce between two concentric conducting spheres with a conically stratified medium is not so simple as that between two concentric conducting spheres with a homogeneous medium.

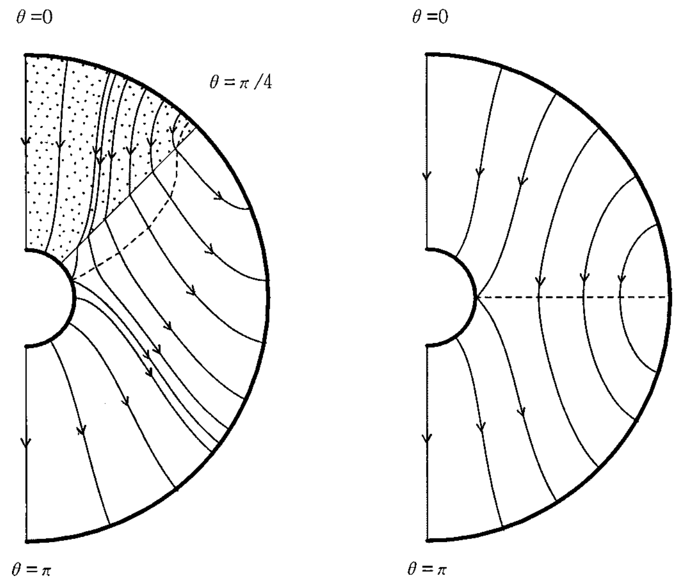


(a)



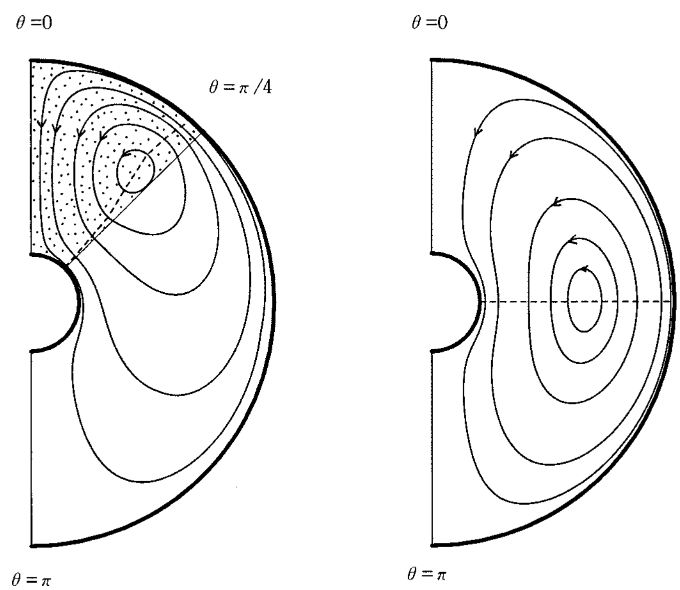
(b)

Fig. 5. (a) Field-distribution pattern of H_θ for the first resonant frequency ($k_0a = 0.69242092$) of a concentric spherical cavity with conically stratified medium when $b/a = 5, \varepsilon_1 = 4, \varepsilon_2 = 1$, and $\alpha = \pi/4$. (b) Field-distribution pattern of H_θ for the first resonant frequency ($k_0a = 0.93728100$) of a concentric spherical cavity with homogenous medium when $b/a = 5$ and $\varepsilon_r = 1$. Throughout, x-, y-, and z-axes represent r, θ, and H_θ (relative value), respectively.



(a) (b)

Fig. 6. (a) Electric-field lines of the TM field for the first resonant frequency ($k_0a = 0.40584501$) of a concentric spherical cavity with conically stratified medium when $b/a = 5, \varepsilon_1 = 4, \varepsilon_2 = 1$, and $\alpha = \pi/4$. (b) Electric-field lines of the TM field for the first resonant frequency ($k_0a = 0.52397273$) of a concentric spherical cavity with homogenous medium when $b/a = 5$ and $\varepsilon_r = 1$. Throughout, the broken lines indicate the locus of points where the radial component of the electric field vanishes.



(a) (b)

Fig. 7. (a) Magnetic-field lines of the TE field for the first resonant frequency ($k_0a = 0.69242092$) of a concentric spherical cavity with conically stratified medium when $b/a = 5, \varepsilon_1 = 4, \varepsilon_2 = 1$, and $\alpha = \pi/4$. (b) Magnetic-field lines of the TE field for the first resonant frequency ($k_0a = 0.93728100$) of a concentric spherical cavity with homogeneous medium when $b/a = 5$ and $\varepsilon_r = 1$. Throughout, the broken lines indicate the locus of the points where the radial component of magnetic field vanishes.

V. Conclusion

In this paper, we have presented an analysis on electromagnetic fields of a cavity formed by two concentric conducting spheres filled with a conically stratified medium. Formulating the field problem using θ-transmission field equations and radially expanding eigenfunctions, we obtained the resonant frequencies and their respective fields in series form. Results have proven that the resonant mode fields in the cavity filled with a conically stratified medium could be expressed in terms of about 15 radially expanded eigenfunctions with the accuracy to at least three digits. Further, we indicated the forms of the typical electric- and magnetic-field lines for the first resonant TM and TE modes in cases of both conically stratified medium and homogeneous medium, and also showed the distribution pattern of the angular field components on the azimuthal coordinate surface. We observed from the patterns of the field distribution that the electromagnetic fields are bounded by a conical dielectric and evanescent toward the surface $\theta = \pi$. Moreover, we found that the locus of the points, where the radial components of the electromagnetic field vanish, is not a straight line in the case of a cavity with a conically stratified medium, while it is a straight line in the case of a cavity with a homogeneous medium. These results mean that the bounce of the electromagnetic fields between two concentric conducting spheres with conically stratified medium is not a simple bounce, as in the case of two concentric conducting spheres with a homogeneous medium.

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